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Dynamic mechanical analysis
– theoretical considerations on mechanical properties

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English summary

Dynamic mechanical analysis (DMA) is a technique used to study and characterize materials. In the test method, small test specimens, or beams, are typically subjected to harmonic, three-point bending loads. During the test programme, the temperature is varied, for a given frequency of the external load, and a set of mechanical properties and the glass transition temperature are determined by the DMA instrument.

The main purpose of the present report is to gain more insight into some of the various material parameters included in DMA and the relation between them. The technical manual for the DMA 2980 instrument from TA Instruments contains a set of expressions for certain stiffness parameters of the materials tested. In the present report, some of the terms from these relations have been investigated carefully. Focus has also been put on increasing the basic knowledge and physical understanding of the storage modulus and the loss modulus obtained through DMA.

In this report, a careful load-displacement analysis of a beam subjected to three-point bending has been conducted. Based on the analysis, it is found that the specific terms in the stiffness relations from the manual represent the effect of deflection due to bending moments and shear forces. Moreover, from a fundamental viscoelastic analysis of the beam, it has been shown that the storage modulus and the loss modulus, both determined through DMA, equal the Young's modulus and the viscosity coefficient, respectively, for isotropic materials.

Sammendrag

Dynamisk mekanisk analyse (DMA) er en testmetode som benyttes for karakterisering av materialer. I testmetoden brukes typisk små prøvestykker/bjelker som utsettes for tre-punkts bøyning, med harmonisk varierende laster. I løpet av et testprogram varieres temperaturen, for en gitt lastfrekvens, og ved hjelp av DMA-instrumentet kan en da blant annet bestemme materialets mekaniske egenskaper og glassovergangstemperatur.

Målet med rapporten er å få inngående kjennskap til noen av de forskjellige parametrene som er inkludert i analysen, og relasjoner mellom disse. Ved FFI benyttes et DMA 2980-instrument fra TA Instruments, og den tekniske manualen for instrumentet inneholder noen uttrykk og relasjoner for relevante stivhetskoeffisienter for materialene som testes. I denne studien ønsker vi å se nærmere på noen av leddene i disse stivhetsrelasjonene. Samtidig er det et mål å få økt kjennskap til og forståelse for den fysiske tolkningen av lagringsmodul og tapsmodul, som er sentrale begreper innen DMA.

I rapporten er det utført en detaljert last-forskyvningsanalyse av en bjelke utsatt for tre-punkts bøyning. Det er vist at de spesifikke leddene i de aktuelle stivhetsrelasjonene representerer effekten av nedbøyning som følge av bøyemomenter og skjærkrefter. Fra en grunnleggende viskoelastisk analyse av bjelken finner en også ut at prøvestykkets lagringsmodul og tapsmodul, som begge kan bestemmes fra DMA, i isotrope tilfeller er identiske med henholdsvis E-modul og viskositetskoeffisient for materialet.

Contents

1 Introduction

Dynamic mechanical analysis (DMA) is a technique used to study and characterize materials. A harmonic load is applied to the test specimen of consideration, and the corresponding deflection, or displacement, is measured. Based on the relationship between the applied load and the observed deflection, the elastic and viscoelastic properties of the material are determined. Moreover, when analysing materials using DMA, the temperature is varied for a given frequency of the applied load. From this approach one can determine the glass transition temperature (T_g) of the material.

At FFI, DMA is applied as an important instrument for the material characterization of various materials, such as polymers, composites and propellants. The tests are carried out using a DMA 2980 from TA Instruments. The main purpose of the present report is to gain more insight into theoretical considerations behind the mechanical properties that are measured using the DMA. We also aim at clarifying the origin of different terms in certain stiffness relations in the technical manual from the manufacturer, as well as their physical meaning. Moreover, it is of interest to get a better understanding on how the storage modulus and the loss modulus, obtained through DMA, are related to the Young's modulus for elastic media.

The report is organized as follows: In Section 2, the specific formula from the technical manual [1] is analysed. The relationship between the material modulus obtained through DMA and Young's modulus is discussed in Section 3. The report ends up with some concluding remarks in Section 4.

2 Analysis of formula for three-point bending test

In the context of DMA, three-point bending is one of the most common test configurations. A simply supported beam is subjected to a vertical load, F, at the midpoint, resulting in a midpoint deflection, δ, see Figure 2.1.

In Section 6 of the manual provided by TA Instruments [1], an equation describing the relationship between the elastic modulus and a set of other parameters for the test beam in three-point bending is offered,

$$E = K_S \frac{L^3}{6I}\left[1+\frac{12}{5}(1+\nu)\left(\frac{t}{L}\right)^2\right]. \tag{2.1}$$

Here, E is stated to be the elastic modulus, L is one half of the beam length, t is the thickness of the beam, I is the second moment of area (for the cross-section of the beam), ν is the Poisson's ratio, and K_S is said to be the measured stiffness. When running DMA tests using the three-point bending configuration, (2.1) is one of the relations applied by the instrument's software to obtain the results.

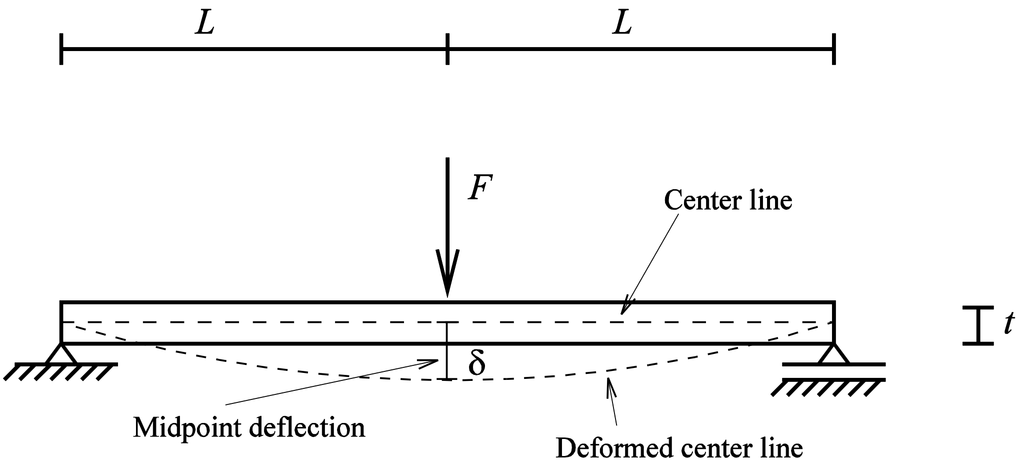

Figure 2.1 Three-point bending test of a beam. The applied load, F , results in a midpoint deflection, δ .

Most of the physical parameters in the expression above are clearly defined. However, the manual does not contain any explanation on the exact meaning of the measured stiffness, K_S . In the context of experimental testing within mechanics, a stiffness parameter is usually interpreted as the (stiffness) coefficient in the relation between an applied or measured load and a corresponding displacement. For the three-point bending test conducted in the DMA instrument (see Figure 2.1), this indicates that K_S might be the stiffness parameter in the relation between the applied load, F , and the resulting midpoint deflection, δ , which reads

$$F = K_S \delta. \tag{2.2}$$

Based on this fundamental assumption, we will now study the origin and meaning of the different terms in (2.1) using linear beam theory.

2.1 Deformation due to bending moments

First, we analyse static deformations due to bending moments for a simply supported beam subjected to a centred load. We assume that the beam is made of an isotropic and linear elastic material. From Euler-Bernoulli beam theory, see e.g. [2;3], it is well known that the relationship between the applied force and the corresponding deflection for a simply supported beam, as shown in Figure 2.1, is given by

$$F = \frac{6EI}{L^3}\delta_{b,} \tag{2.3}$$

where δ_b is the part of the midpoint deflection that is due to bending moments. Expressing (2.3) with respect to the deflection, we obtain

$$\delta_b = \frac{FL^3}{6EI}. \tag{2.4}$$

Now, dividing equation (2.3) by δ_b, assuming that the entire midpoint deflection of the beam is represented by δ_b, and introducing the measured stiffness, K_S, from (2.2), we obtain

$$K_S = \frac{6EI}{L^3}, \tag{2.5}$$

which after some simple rearrangements reads

$$E = \frac{K_S L^3}{6I}. \tag{2.6}$$

By careful inspection, it is seen that the right hand side of (2.6) is equal to the first term on the right hand side of (2.1). Therefore, it can be concluded that this term is due to bending.

2.2 Deformation due to shear forces

Although bending moments in most cases are the main contributor to deflections of beams, shear forces may also play a crucial role in certain cases, in particular if the length-to-thickness ratio L/t is not much larger than unity. In the following, we will therefore study the shear deformations of the beam shown in Figure 2.1, and investigate whether the second term on the right hand side of (2.1) is related to shear forces, or not.

From simple equilibrium considerations of the beam subjected to three-point bending, it is found that the distribution of (vertical) shear forces, Q, along the beam is as shown in the shear force diagram depicted in Figure 2.2. The governing differential equation for shear deformations of beams may be derived from the basic relationship between the shear stress and shear strain.

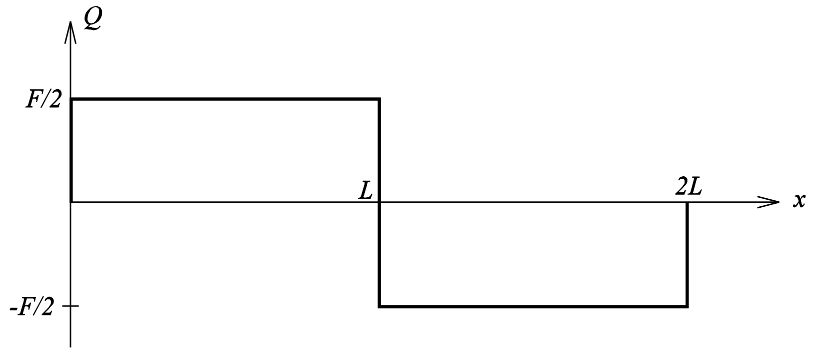

Figure 2.2 Shear force (Q) diagram for the simply supported beam subjected to three-point bending.

Assuming a linear elastic material and deformations in the (vertical) $xy-$plane, as indicated in Figure 2.3, which shows the shear forces and deformation pattern of an infinitesimal element, dx, of the left part of the beam, i.e. $x < L$, we obtain

$$\tau_{xy} = G\gamma_{xy}. \tag{2.7}$$

Here, τ_{xy} is the shear stress, γ_{xy} is the shear strain, and G is the shear modulus of the material. Using the definition of the shear strain, (2.7) can be expressed as

$$\frac{dv}{dx} = \frac{1}{G}\tau_{xy}, \tag{2.8}$$

where v is the displacement component in the vertical direction.

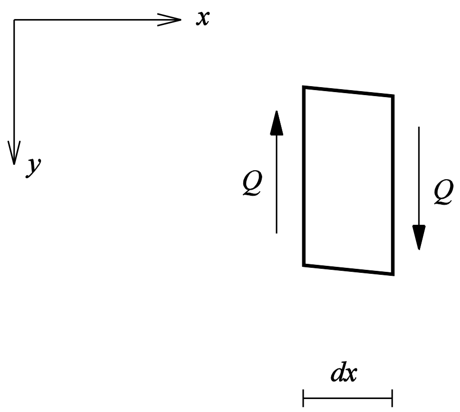

Figure 2.3   Shear forces and deformation pattern of an infinitesimal element, dx, of the beam.

It is well-known that the shear stress distribution at a given cross-section along the beam is not constant. For a rectangular cross-section, the shear stress varies as a parabolic function [2]. The stress value is zero at the top and bottom of the cross-section (due to free surfaces), while the maximum value is attained at the mid-thickness level, see Figure 2.4. However, when analysing the deflection of the beam due to shear, a uniform approximation for the shear stress distribution is usually adopted. A natural choice for this approximation is to apply the mean shear stress, given by

$$\tau_{xy,mean} = \frac{Q}{A}, \tag{2.9}$$

where Q is the shear force, and A is the area of the cross-section. However, more detailed analysis methods, including energy considerations, suggest a modified approximation for the shear stress to be used [2],

$$\overline{\tau_{xy}} = \mu\tau_{xy,mean} = \mu\frac{Q}{A}, \tag{2.10}$$

where μ is a shear coefficient. For rectangular cross-sections, in particular, it can be shown that the optimal choice is $\mu = 6/5$. Inserting (2.10) into (2.8) yields

$$\frac{dv}{dx} = \mu \frac{Q}{GA}, \tag{2.11}$$

where GA/μ is denoted the effective shear stiffness.

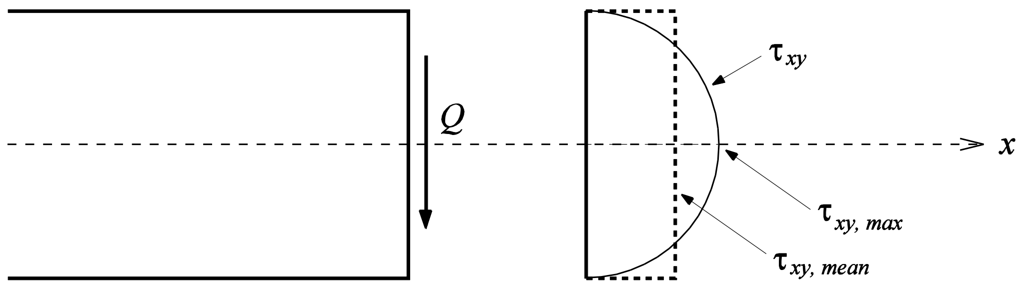

Figure 2.4 Shear stress distribution for a beam with a rectangular cross-section.

In this study, we are interested in the deflection at the midpoint, i.e. at $x = L$, of the beam shown in Figure 2.1. The vertical displacement due to shear can be determined from the boundary value problem, which, after introducing $Q = F/2$ (from Figure 2.2), reads

$$\frac{dv}{dx} = \mu \frac{F}{2GA}, \quad x \in (0, L) \tag{2.12}$$

$$v(x=0) = 0. \tag{2.13}$$

The displacement field is obtained by solving the differential equation in (2.12), which in this case is done through simple integration, and then applying the boundary condition in (2.13) to determine the integration constant. After calculations we get

$$v(x) = \mu \frac{F}{2GA} x. \tag{2.14}$$

The deflection at the midpoint of the beam is obtained by evaluating the displacement field (2.14) at $x = L$,

$$\delta_s = v(x=L) = \mu \frac{FL}{2GA}. \tag{2.15}$$

Finally, inserting $A = bt$ (where b is the width of the beam), $G = E/\left(2(1+\nu)\right)$, which is valid for isotropic, linear elastic materials, and putting in the optimal value for the shear coefficient for a beam with a rectangular cross-section, we obtain

$$\delta_s = \frac{6}{5} \frac{FL(1+\nu)}{Ebt}. \tag{2.16}$$

2.3 Total deformations

With the expressions for the mid-point deflection due to both bending and shear at hand, we are now in a position to express the total deflection. From assuming small deformations and a linear elastic material, the contributions from the deformation due to bending moments and due to shear forces may be summed, i.e. applying the principle of superposition [2]. The total deflection may hence be expressed as the sum of (2.4) and (2.16), reading

$$\begin{aligned}\delta &= \delta_b + \delta_s \\ &= \frac{FL^3}{6EI} + \frac{6}{5}\frac{FL\left(1+\nu\right)}{Ebt} \\ &= \frac{FL^3}{6EI}\left(1 + \frac{36}{5}\frac{\left(1+\nu\right)}{bt}\frac{I}{L^2}\right).\end{aligned} \tag{2.17}$$

By inserting the expression for the second moment of area of the beam with rectangular cross-section, $I = \left(bt^3\right)/12$, into the second term of the parenthesis in (2.17), we obtain

$$\delta = \frac{FL^3}{6EI}\left(1 + \frac{3}{5}\left(1+\nu\right)\left(\frac{t}{L}\right)^2\right). \tag{2.18}$$

The latter relation is now multiplied by E/δ. In addition, the measured stiffness from (2.2) is introduced. This results in the final expression for the elastic modulus

$$E = K_S\,\frac{L^3}{6I}\left(1 + \frac{3}{5}\left(1+\nu\right)\left(\frac{t}{L}\right)^2\right). \tag{2.19}$$

While the first term on the right hand side of (2.19) is identical to the corresponding term in (2.1), there is a small discrepancy for the second term. In (2.19), the numerical factor of this term reads $3/5$, while the corresponding factor in (2.1) is $12/5$. Because all the remaining factors and parameters in these terms agree, we are convinced that the second term of (2.1) is meant to represent the contribution due to shear deformations, and that the only reason for the difference between (2.1) and (2.19) is an error in (at least) one of the analyses leading to the final results.

In order to examine the problem a bit closer, trying to figure out the reason for the differences between (2.1) and (2.19), we have conducted a completely independent analysis of a simply supported beam with length $\hat{L}$ (instead of $2L$ as applied in the above analysis), and still with a vertical midpoint load F. Furthermore, all the remaining parameters are identical to the ones applied previously. The resulting expression for the Young’s modulus now reads

$$E = K_S \frac{\hat{L}^3}{48I}\left(1+\frac{12}{5}(1+\nu)\left(\frac{t}{\hat{L}}\right)^2\right). \tag{2.20}$$

By comparing (2.20) with (2.1) it is seen that the expressions within the outer set of parentheses are identical (except for the use of $\hat{L}$ and L in (2.20) and (2.1), respectively), but now the numerical factors in the denominator to the left of the first parenthesis differ. On the other hand, by substituting $\hat{L} = 2L$ into (2.20) we end up with an expression for the Young's modulus that is identical to (2.19). The fact that the result of this additional analysis completely agrees with (2.19), serves to increase the trustworthiness of the derivations presented in the present report. Therefore, we believe that our results are correct, and that there is an error in the expression (2.1) presented in the technical manual [1]. As discussed above, this error is related to the shear deformations of the beam, which are known to be small compared to the deflections due to bending moments in a wide range of practical applications.

Let us conclude this section by including a brief quantitative discussion on the effects of the shear deformations and the error introduced in the shear term of (2.1). For example, by assuming the thickness-to-length ratio of the beam to be 0.1 (which means that $t/2L = 0.1$), the contribution from the shear deformations is limited to 3%, while the error in the shear term of (2.1) means that the Young's modulus will be overestimated by approximately 10%. Thus, the effect of the error in this equation might be relatively large, even though the geometric properties of the beam indicate that the effects due to shear are small.

3 Storage modulus and loss modulus

As indicated at the beginning of this report, DMA is known to be an experimental technique that is well suited for studying the mechanical behaviour of viscoelastic materials such as polymers. In DMA, the most common test specimen for such materials is a short beam subjected to three-point bending. A harmonic load is typically applied, and the time-dependent deflection, including the amplitude and the phase lag, is registered by the instrument. Based on these response quantities, the storage modulus and the loss modulus of the material are calculated. Furthermore, by varying the temperature of the specimen during loading, the glass transition temperature of the material may also be obtained.

In this part of the report, we will analyse the time-dependent behaviour of the beam subjected to three-point bending. By introducing the relations derived in Section 2, the physical interpretation of the storage modulus and the loss modulus will be examined.

Assuming that the behaviour of the viscoelastic material of consideration may be described by a Kelvin-Voigt model, the normal stress (σ) can be expressed as

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}, \tag{3.1}$$

where E is the Young's modulus, ε is the normal strain, η is the viscosity coefficient, and $\dot{\varepsilon}$ is the rate of change of the normal strain, i.e. $\dot{\varepsilon} = d\varepsilon / dt$. The first term on the right hand side of (3.1) represents the elastic part of the material behaviour, whereas the second term models the viscous part.

As we have seen in Section 2, based on the (quasi-) static load-displacement relation (2.2) and the assumption of a linear elastic material behaviour, we were able to derive equation (2.19) relating the measured stiffness parameter, K_S , to certain elastic and geometric properties of the beam. Imagine a beam made of a pure viscous material. Then there is a one-to-one relationship between the applied load and the rate of change of displacement, or the velocity. In the case of three-point bending, this relation may be written

$$F = C_S \dot{\delta}, \tag{3.2}$$

where F is the applied force, C_S is the viscous stiffness parameter for the beam, and $\dot{\delta}$ is the rate of change of midpoint deflection. By repeating the same kind of analysis as conducted in Section 2 for the present pure viscous beam, we end up with an equation similar to (2.19), in which E and K_S are replaced by the viscous coefficient, η , and the corresponding viscous stiffness, C_S , respectively. That is, a relatively simple relation between C_S , η , and several geometric properties of the beam is established.

Finally, we want to perform a careful analysis of the dynamics of the three-point bending beam aiming at deriving relations between the stiffness parameters, K_S and C_S , and response properties that are directly accessible to the DMA instrument. We consider a beam made of a viscoelastic material, described by the Kelvin-Voigt model (3.1). As stated in Section 2, test specimens applied in DMA are typically subjected to harmonic loads. By neglecting inertia forces, which are assumed to be small compared to the elastic and viscous terms, and including the remaining load terms introduced through (2.2) and (3.2), the dynamic equilibrium equation for the midpoint deflection of the beam can be expressed by [4]

$$C_S \dot{\delta}(t) + K_S \delta(t) = F_0 \sin(\omega t), \tag{3.3}$$

where F_0 is the amplitude of the applied load, and ω is the angular frequency of the load. This is an inhomogeneous differential equation, which means that the general solution consists of the sum of the (general) homogeneous solution and the particular solution. The amplitude of the homogeneous part will be reduced due to the viscous damping of the material, and the particular solution will thus be the dominating part. Based on the form of the differential equation and the load function, a natural guess for the particular solution is given by

$$\delta_p(t) = D_1 \cos(\omega t) + D_2 \sin(\omega t), \tag{3.4}$$

where D_1 and D_2 are constants to be determined. By inserting (3.4) into (3.3), and equating the sine and cosine terms separately, we obtain a system of equations for the constants, which may be written,

$$\begin{aligned} -C_S D_1 + K_S D_2 &= F_0 \\ K_S D_1 + C_S D_2 &= 0, \end{aligned} \tag{3.5}$$

with the unique solutions

$$\begin{aligned} D_1 &= -\frac{F_0 C_S}{C_S^2 + K_S^2} \\ D_2 &= \frac{F_0 K_S}{C_S^2 + K_S^2}. \end{aligned} \tag{3.6}$$

Thus, the particular solution of (3.3) is defined through (3.4) and (3.6). As pointed out above, the amplitude and the phase lag of the midpoint deflection is monitored by the DMA instrument during loading. Therefore, for the present analysis it is advantageous to rewrite the particular solution in the form

$$\delta_p(t) = R\sin(\omega t - \varphi), \tag{3.7}$$

where R is the amplitude and φ is the phase lag. By using a formula for the sine function of the difference between two angles (or arguments) and some simple algebraic manipulations, it may be shown that

$$\begin{aligned} R &= \frac{F_0}{\sqrt{C_S^2 + K_S^2}} \\ \varphi &= \arctan\left(\frac{C_S}{K_S}\right). \end{aligned} \tag{3.8}$$

The fact that the phase angle is close to zero in problems dominated by elastic forces ($K_S \gg C_S$), and that the phase lag approaching $\pi/2$ (or 90°) in viscous-dominated problems ($C_S \gg K_S$), are easily derived from (3.8). By inverting the relations in (3.8), we obtain

$$\begin{aligned} K_S &= \frac{F_0 \cos\varphi}{R} \\ C_S &= \frac{F_0 \sin\varphi}{R}, \end{aligned} \tag{3.9}$$

which states an important set of results. Based on the amplitude of the applied load and the resulting amplitude and phase lag of the midpoint deflection, the elastic and viscous stiffness parameters included in the governing dynamic equilibrium equation for the beam can be calculated by the DMA instrument. The storage modulus, which is identical to the Young's modulus, E, for isotropic Kelvin-Voigt materials, is then predicted from (2.19) and (3.9). Furthermore, the loss modulus, or the viscous coefficient, η, is obtained from (3.9) and a relation between η and C_S, which can be derived in a similar manner as the analysis that led to (2.19).

4 Summary

In this report, a careful load-displacement analysis of a beam subjected three-point bending has been conducted. Based on this analysis, we have found that the specific terms in the stiffness relations in the technical manual for the DMA manual represent deflection due to bending moments and shear forces, respectively. Moreover, from a fundamental viscoelastic analysis of the beam, it has been shown that the storage modulus and the loss modulus, both determined through DMA, are identical to the Young's modulus and the viscosity coefficient, respectively, for isotropic materials.

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